Ribenboim's generalized power series and weighted Rota-Baxter categories

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Rota-Baxter Algebra

The Rota-Baxter equation is typically presented as capturing the integration by parts rule in analogy to the Leibniz rule in differential algebra. (This is historically wrong.)

Definition

A **differential algebra** *is a (commutative) algebra R with a map* $d\colon R \to R$ *satisfying* $\quad d(xy) = xd(y) + d(x)y.$

Take a suitably differentiable class of functions, e.g. a polynomial ring, as an example.

For integral calculus, we have a less well-known structure:

Rota-Baxter Algebra II

Definition

A **Rota-Baxter algebra** is a (commutative) algebra R with a map $P\colon R \to R$ satisfying the **Rota-Baxter equation:**

$$P(x)P(y) = P(xP(y)) + P(P(x)y)$$

.

This captures the integration by parts formula as seen by taking R to be a suitable class of functions on $\mathbb{R}$ and defining

$$P(f)(x) = \int_0^x f(t)dt$$

See the work of Li Guo, especially his book *Introduction to Rota-Baxter Algebras*.

Rota-Baxter Algebra III: More Examples

- Consider $\mathbb{R}[x]$ with multiplication given by $x^m \cdot x^n = \binom{m+n}{n} x^{m+n}$. Then $P(x^n) = x^{n+1}$ is a Rota-Baxter algebra.

Now let X be a set and X^* be the free monoid on X. If w,w' are elements of X^*, i.e. words in X, then a *shuffle* of w_1 and w_2 is a permutation of ww' such that the internal orders of w_1 and w_2 are preserved.

Let $A = k[X^*]$ and define $m : A \otimes A \to A$ by

$$m(w_1 \otimes w_2) = \sum \{w | w \text{ is a shuffle of } w_1 \text{ and } w_2\}$$

This determines a commutative, associative, unital algebra structure on A.

Rota-Baxter Algebra IV: The Shuffle Example

Let V be an arbitrary k-vector space. Let X be a basis for V. Then $k[X^*] \cong T(V) = k \oplus V \oplus V \otimes V \ldots$,

$T(V)$ is usually viewed as the free (non-commutative) algebra on V. But here we equip it with the shuffle algebra multiplication.

Theorem

With the setup described above, if $v \in V$, we have a Rota-Baxter operator $P_v \colon T(V) \to T(V)$ defined by $P_v(w) = v \otimes w$.

Rota-Baxter Algebra V: Generalization

The definition of Rota-Baxter algebra can be generalized to include weights. (In fact, this was the original form of the equation.)

Definition

A be a k-algebra. A is a *Rota-Baxter algebra or RB-algebra*[a]*. of weight* λ if equipped with a k-linear map $P\colon A \to A$ such that for all $x, y \in A$

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy)$$

[a] RB stands for Rota-Baxter, I swear.

There is a similar weighted version of the notion of differential algebra.

Rota-Baxter Algebra VI: More Examples

- Let R be a commutative ring. Let $A = \Pi_{i=0}^{\infty} R$ be a countable product of the ring with pointwise operations. Define $P\colon A \to A$ by $P(a_1, a_2, a_3, \ldots) = (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots)$. Then P is an RB-operator of weight 1.
- Let $R = k[t]$ and suppose $q \in k$ is not a root of unity. Then for $n \geq 1$

$$P(t^n) = \frac{q^n}{1 - q^n} t^n \qquad \text{is an RB-operator of weight 1.}$$

The intuition here is that this is really the operator:

$$P(f)(t) = \sum_{k=1}^{\infty} f(q^k t)$$

This is also called the *q-integral*.

Rota-Baxter Algebra VII: More Examples

- Similarly, consider a subalgebra of $Cont(\mathbb{R})$ closed under the following operation:

$$P(f)(x) = -\lambda \sum_{n=0}^{\infty} f(x + n\lambda)$$

Then P is an RB-operator of weight λ.

- Let $M_n^u(k)$ be the algebra of upper-triangular $n \times n$ matrices. Define

$$P([a_{ij}]) = [b_{ij}] \quad \text{where} \quad b_{ij} = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad b_{ij} = \sum_{j=1}^{n} a_{ij} \quad \text{otherwise}$$

In words, P adds the elements of a row and puts the sum on the corresponding diagonal. Then P is an RB-operator of weight -1.

Rota-Baxter Algebra VIII: Laurent Series

Consider the algebra of Laurent series $R = \{\sum_{n=m}^{\infty} a_n t^n | a_n \in k\}$. Then define

$$P(\sum_{n=m}^{\infty} a_n t^n) = \sum_{n=m}^{-1} a_n t^n$$

Then P is an RB-operator of weight -1. Similarly if

$$P'(\sum_{n=m}^{\infty} a_n t^n) = \sum_{n=0}^{\infty} a_n t^n \quad \text{then } P' \text{ is an RB-operator of weight -1}$$

These examples are the inspiration for the use of RB-algebras in the theory of renormalization in quantum field theory.

Ribenboim's generalized power series

We'll need the following technical condition:
Let $(M, +, \leq)$ be a partially ordered (commutative) monoid. M is *strictly ordered* if

$$s < s' \implies s + t < s' + t \ \forall s, s', t \in M \ .$$

We will henceforth assume that all the monoids we work with are strictly ordered.

Definition

An ordered monoid is *artinian* if all strictly descending chains are finite; that is, if any list $(m_1 > m_2 > \cdots)$ must be finite. It is *narrow* if all discrete subsets are finite; that is, if all subsets of elements mutually unrelated by $\leq$ must be finite.

Ribenboim's generalized power series II

Definition

Let V be a vector space, and recall that the *support* of a function $f : M \to V$ is defined by $supp(f) = \{m \in M | f(m) \neq 0\}$. Define the *space of Ribenboim power series from* M *with coefficients in* V, $G(M, V)$ to be the set of functions $f : M \to V$ whose support is artinian and narrow.

If A is also a commutative $\mathbb{K}$-algebra, then $G(M, A)$ is a commutative $\mathbb{K}$-algebra with

$$(f \cdot g)(m) = \sum_{(u,v) \in X_m(f,g)} f(u) \cdot g(v)$$

where

$$X_m(f, g) := \{(u, v) \in M \times M | u + v = m \text{ and } f(u) \neq 0, g(v) \neq 0\}$$

Ribenboim's generalized power series III

This requires the following observation. It is where the strictness property gets used:

Proposition

The set $X_m(f, g)$ is finite for $f, g \in G(M, V)$.

There are lots of examples.

- Let $M = \mathbb{N}$. The result is the usual ring of power series with coefficients in A.
- Let $M = \mathbb{Z}$. The result is the ring of Laurent series with coefficients in A.

Ribenboim's generalized power series IV: More examples

- Let $M = \mathbb{N}^n$, with pointwise order. The result is the usual ring of power series in n-variables with coefficients in A.

  This example is due to Ribenboim and was his motivation:

- Let $M = \mathbb{N}\backslash\{0\}$ with the operation of multiplication, equipped with the usual ordering. Then $G(M, \mathbb{R})$ is the ring of arithmetic functions (i.e. functions from the positive integers to the complex numbers), and multiplication is Dirichlet's convolution:

$$(f \star g)(n) = \sum_{d|n} f(d)g(\frac{n}{d})$$

Ribenboim's generalized power series V: Rota-Baxter structure

It is an observation of Guo and Liu that these rings frequently have a weighted Rota-Baxter operator:
Let M be any strictly ordered monoid, and let M_1, M_2 be such that the disjoint union $M_1 \dot{\cup} M_2 = M$. Define a map $\Pi_V \colon G(M, A) \to G(M, A)$ by

$$\Pi(f)(m) = \begin{cases} f(m) & \text{if } m \in M_1 \\ 0 & \text{if } m \in M_2 \end{cases}$$

Theorem (Guo,Liu)

The operator Π makes $G(M, A)$ into a RB-algebra of weight -1 if and only if M_1 and M_2 are subsemigroups of M.

Modalities: Back to weight 0

We have the following slight generalizations of module with differentiation and module with integration.

Definition

Let A be a commutative $\mathbb{K}$-algebra, and let M be a right A-module. Then M is a *module with differentiation* if it is equipped with a map $\partial \colon A \to M$ such that $\forall a, b, \in A$ we have

$$\partial(ab) = \partial(a) \cdot b + \partial(b) \cdot a$$

Let A be a commutative $\mathbb{K}$-algebra, and let M be a right A-module. M is a *module with integration* if it is equipped with a map $\pi \colon M \to A$ such that $\forall m, n \in M$ we have

$$\pi(m)\pi(n) = \pi(m \cdot \pi(n)) + \pi(\pi(m) \cdot n) \ .$$

Modalities & scalars

Definition

- Let $\mathbb{C}$ be a (symmetric) monoidal category. The monoid of *scalars* for $\mathbb{C}$ is the set $\mathrm{Hom}_{\mathbb{C}}(I, I)$. In an additive category, it's a commutative ring. This monoid (or ring) acts on all hom-sets using the unit isomorphisms.

- An *algebra modality* on $\mathbb{C}$ is a monad T such that every object $T(V)$ has a (commutative) algebra structure and various coherence equations hold.

- A *module modality for T* is a functor H such that every $H(V)$ is a $T(V)$-module and various coherence conditions hold.

Leibniz and RB-categories (of weight 0)

Definition

A *Leibniz category* is a monoidal category with an algebra modality T and a module modality H and a natural transformation $\partial\colon T \to H$ which gives a module with differentiation for all V.

Every codifferential category is Leibniz with $H(V) = T(V) \otimes V$, which is the free $T(V)$-module on V. Kähler categories (RB,Cockett,Porter,Seely) are Leibniz categories with a universal property.

Definition

An *RB-category* is a monoidal category with an algebra modality T and a module modality H and a natural transformation $\pi\colon H \to T$ which gives a module with integration for all V.

Every cointegral category is Leibniz with $H(V) = T(V) \otimes V$.

Leibniz and RB-categories (of arbitrary weight)

Consider the RB-equation again:

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy)$$

If we are going to set up an analogous equation using the map $\pi \colon H \to T$, we need a multiplication in $H(A)$ as well as $T(A)$. This leads to the following:

Definition

Let (T, m_T, e_T) be an algebra modality. Then an algebra modality (H, m_H, e_H) is a *module algebra modality associated to* T if $H(A)$ is a TA-module for every object A; that is, if there is a map $\bullet \colon H(A) \otimes T(A) \to H(A)$ such that various equations hold.

RB-categories of arbitrary weight

- Every algebra modality is a module algebra modality after itself.

Definition

Let $\mathcal{C}$ be a symmetric monoidal closed category equipped with an algebra modality T and let H be a module algebra modality associated to T A *Rota-Baxter transformation of weight* λ for a given scalar λ is a natural transformation $\Pi \colon H \to \; T$ such that the following equation holds:

$$(\Pi \otimes \Pi); m^T = (id \otimes \Pi); \bullet; \Pi + (\Pi \otimes id); \sigma; \bullet; \Pi + m^H; \lambda\Pi$$

Equipped with such a Π, $\mathcal{C}$ is called a *Rota-Baxter category of weight* λ.

Ribenboim's generalized power series: Categorical structure

The Ribenboim theory fits nicely into the differential/integral category framework, but with weights. In particular, we get algebra modalities. We consider $G(M, -)$ as a functor $Vec \to Vec$.

Proposition

$G(M, -)$ is a monad, with monadic unit $\eta_V \colon V \to G(M, V)$ given by

$$\eta \colon v \mapsto \left[m \mapsto \begin{cases} v & \textit{if } m = 0_M \\ 0_V & \textit{if } m \neq 0_M \end{cases} \right]$$

and monadic multiplication $\mu_V \colon G(M, G(M, V)) \to G(M, V)$ given by

$$\mu \colon h \mapsto \left[m \mapsto \sum_{(u,v) \in X_m(h)} h(u)(v) \right]$$

Ribenboim's generalized power series VII: More categorical structure

But to have an algebra modality, we need more than vector spaces in the second coordinate:

Lemma

There is a distributive law of monads of $G(M, -)$ over S, the symmetric algebra monad.

Corollary

The composite functor $G(M, S-)$ is a monad on Vec.

Ribenboim's generalized power series VII: More categorical structure

Proposition

The monad $G(M, S-)$ is an algebra modality.

To show this works, we use:

Lemma (RB, Lucyshyn-Wright, O'Neill)

Let S be the symmetric algebra monad. Then the commutative algebra modalities on a category are in bijective correspondence with pairs (T, ψ) where T is a monad and $\psi\colon S \to T$ is morphism of monads.

It's an example

Theorem

Let M be any strictly ordered monoid, and let M_1, M_2 be subsemigroups such that the disjoint union $M_1 \dot{\cup} M_2 = M$. Define a map $\Pi_V \colon G(M, SV) \to G(M, SV)$ by

$$\Pi(f)(m) = \begin{cases} f(m) & \text{if } m \in M_1 \\ 0 & \text{if } m \in M_2 \end{cases}$$

This defines a Rota-Baxter category of weight -1 where $G(M, SV)$ is viewed as a module algebra over itself.

Going forward

Lately, we've been looking at categorical aspects of these generalized power series, which don't seem to have been much explored after Ribenboim's original work.

Today, I'll talk about **Morita equivalence**.

Definition

Two rings R and S are *Morita equivalent* if their categories of (left) modules are equivalent. (We note that the categories of left modules are equivalent if and only if the categories of right modules are equivalent.) We denote Morita equivalence by $R \approx S$.

Morita Equivalence: Examples

- Two commutative rings are Morita equivalent if and only if they are isomorphic.
- For any ring R, we have $R \approx M_n(R)$.

These were both well-known before Morita's work. Morita rephrased equivalence in terms of bimodules, which has allowed the ideas to be generalized via bicategories.

Morita's Theorem

Theorem

Suppose R and S are rings and $R \approx S$. Let $F\colon R-\text{Mod} \to S-\text{Mod}$ and $G\colon S-\text{Mod} \to R-\text{Mod}$ be functors inducing an equivalence. Then letting $P = F(R)$ and $Q = G(S)$, then

- $P = {}_SP_R$ *and* $Q = {}_RP_S$ *are faithfully balanced bimodules.*
- ${}_SP_R \cong Hom_S(Q, S) \cong Hom_R(Q, R)$
- ${}_RQ_S \cong Hom_S(P, S) \cong Hom_R(P, R)$
- $F \cong P \otimes_R -$
- $G \cong Q \otimes_S -$

Morita contexts I

Let C be a ring, let $e \in C$ be an idempotent and let $e' = 1 - e$ be its complementary idempotent. Then we obtain a decomposition of C as

$$C \cong e'Ce' \oplus e'Ce \oplus eCe' \oplus eCe$$

We arrange these into a 2×2 matrix as:

$$\begin{bmatrix} e'Ce' & e'Ce \\ eCe' & eCe \end{bmatrix}$$

We rename the entries of this matrix as follows:

$$\begin{bmatrix} B & M \\ N & A \end{bmatrix}$$

We have $M =_B M_A$ and that $N =_A N_B$. We have bimodule maps as follows:

$$f : M \otimes_A N \to B \qquad g : N \otimes_B M \to A$$

Morita contexts II

From the associativity of the multiplication in C, we conclude, for all $n_i \in N$ and $m_j \in M$

$$n_1 f(m_2 \otimes n_3) = g(n_1 \otimes m_2)n_3 \qquad f(m_1 \otimes n_2)m_3 = m_1 g(n_2 \otimes m_3)$$

We present the following two equivalent definitions of *Morita context*.

Definition (Version 1)

A *Morita context* between rings A and B consists of a ring C equipped with an idempotent e such that $A \cong eCe$ and $B \cong e'Ce'$ where $e' = 1 - e$.

Morita contexts III

Definition (Version 2)

A *Morita context* between rings A and B consists of bimodules $M = {}_B M_A$ and $N = {}_A N_B$ and bimodule maps

$$n_1 f(m_2 \otimes n_3) = g(n_1 \otimes m_2) n_3 \qquad f(m_1 \otimes n_2) m_3 = m_1 g(n_2 \otimes m_3)$$

satisfying the previous associativity constraints.

Definition

The quotient rings:

$$\overline{C} = C/(e) \qquad \text{and} \qquad \overline{C}' = C/(e')$$

are the *Morita defects* of C.

Morita contexts IV

Theorem

Let (C, e) be a Morita context between A and B. The following are equivalent:

- *A and B are Morita equivalent via (C, e).*
- *Both Morita defects are 0.*
- *The maps f and g above are isomorphisms.*

One can form the Morita context ring in any additive category. It is always a ring. These are called *formal matrix rings*.

This approach to Morita equivalence generalizes to most monoidal settings. B. Pecsi has given a definition of Morita context in terms of arbitrary bicategories. There are lots of interesting examples.

Abstract Morita theory

One can consider Morita theory in a number of settings.

- For inverse semigroups, there's been a great deal of work, by Talwar, Funk, Lawson, Steinberg and others. Morita equivalence can be framed in terms of *equivalence bimodules* or *enlargements*. There is also a topos-theoretic interpretation of Morita theory. (Two inverse semigroups are equivalent if certain categories of presheaves are equivalent.) Funk, Lawson and Steinberg show all the approaches are equivalent.

- For C^*-algebras, Morita equivalence can be expressed in terms of *imprimitivity bimodules* (Rieffel).

S-Posets

In particular, we can look at partially ordered monoids (pomonoids).

Definition

- If S is a pomonoid, then a poset P is an *S-poset* if equipped with a (right) action of S such that the resulting map $\cdot \colon P \times S \to P$ is monotone in each variable.
- If S, T are pomonoids, then a poset P is an *$S - T$-biposet* if equipped with a left action of S and a right action of T such that the resulting map $S \times P \times T \to P$ is monotone in each variable and $(sp)t = s(pt)$ for all $s \in S, p \in P, t \in T$.
- A *morphism* of S-posets is a monotone map which commutes with the S-action.
- We thus get a category denoted Pos_S.

Theorem

The category Pos_S is complete, cocomplete and cartesian closed.

Tensor product of S-posets

Given two biposets ${}_SM_T$ and ${}_TN_R$ for pomonoids S, T and R, one constructs the tensor product ${}_SM \otimes_T N_R$ in the evident way. One considers the cartesian product $M \times N$. Then consider the congruence generated by the set

$$\{((mt, n), (m, tn)) | m \in M, n \in N, t \in T\}$$

One can see that $m \otimes n \leq m' \otimes n'$ if and only if there exist elements $m_i \in M, n_i \in N, u_i, v_i \in T$ such that we have the following scheme:

$$\begin{array}{rcr} m \leq m_1 u_1 & & \\ m_1 v_1 \leq m_2 u_2 & & u_1 n \leq v_1 n_2 \\ & \vdots & \\ m_k v_k \leq m' & & u_k n_k \leq v_k n' \end{array}$$

The result will be an $S - R$ bimodule satisfying the usual properties.

Morita theory for pomonoids

The following results are due to Laan. They are typical of the general structure

Theorem

Let $F\colon \mathrm{Pos}_S \to \mathrm{Pos}_T$ be a Pos*-equivalence. Then*

- *There exists $P = {}_S P_T$ with $F \cong - \otimes_S P$.*
- *There exists $Q = {}_T Q_S$ with $F \cong Hom_S(Q, -)$ and the inverse of F given by $G \cong - \otimes_T Q$.*
- ${}_T(Q \otimes_S P)_T \cong T$

Morita theory for pomonoids II

A frequent question in all of these theories is what properties are preserved under Morita equivalence?

Theorem

Suppose $F\colon \mathrm{Pos}_S \to \mathrm{Pos}_T$ is a Pos*-equivalence. Let $A = A_S$ be an S-poset. Then*

- *A and $F(A)$ have isomorphic lattices of subobjects.*
- *A and $F(A)$ have isomorphic lattices of congruences.*
- *If A is flat, then so is $F(A)$.*

Analogous results hold for Morita equivalent rings, inverse semigroups, C^*-algebras, etc.

Morita theory for pomonoids III: Boring Theorem

Theorem

Let R and S be Morita equivalent rings. Let P and P' be Morita equivalent pomonoids. Then the rings $G(P, R)$ and $G(P', S)$ are Morita equivalent.

For example, if R and S are related by the bimodule $M = {}_S M_R$, then $G(P, M)$ is a bimodule relating $G(P, R)$ and $G(P, S)$, etc.

Next?

- Hopefully, the converse is false and we get interesting relations between different rings of generalized power series.
- It would be especially interesting if such results arose from a different bicategory than just the usual bicategory of rings and bimodules.
- Fix the requirement of strictness in the definition of $G(M, R)$. Will this involve generalizing away from pomonoids?.
- We could also consider $G(M, A)$ with A a C^*-algebra. Is this a way to relate the various notions?
- Brzezinski, Marquez & Vercruysse reframe Morita equivalence in terms of pairs of monads, Eilenberg-Moore categories and prove a Beck monadicity theorem.