Day convolution, ∞-operads and Goodwillie calculus

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What sort of categories?

$\mathcal{C}$: pointed compactly-generated ∞-category

- ∞-category = quasicategory = weak Kan complex, a model for an $(\infty, 1)$-category (Boardman-Vogt, Joyal, Lurie)

  (think of $\mathcal{C}$ as a category enriched in simplicial sets or topological spaces: for $X, Y \in \mathcal{C}$, there is a mapping space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$)
- compactly-generated = generated under filtered colimits by a small subcategory $\mathcal{C}^{\omega}$ that has finite colimits
- pointed = has a null object $*$ (both initial and terminal)

Examples

- Top_*: pointed topological spaces (or Kan complexes)
- Sp: spectra
- $\mathrm{Alg}_{\mathcal{O}}$: (augmented) algebras in Sp over an ∞-operad $\mathcal{O}$

Spectra

Sp is the stable ∞-category freely generated under colimits by a single object S (the sphere spectrum):

- stable means (fibre sequences = cofibre sequences), (pushouts = pullbacks)
- pointed compactly-generated
- closed symmetric monoidal ∞-category with *smash product* $\wedge$ satisfying $S \wedge S \simeq S$
- analogue: chain complexes of abelian groups under $\otimes$
- the *stabilization* of Top_*: there is a universal adjunction

$$\Sigma^\infty : \mathrm{Top}_* \rightleftarrows \mathrm{Sp} : \Omega^\infty$$

such that any colimit-preserving functor from Top_* to a stable ∞-category factors through Σ^∞

What sort of functors?

Philosophy of this Talk

We can understand an ∞-category $\mathcal{C}$ by studying the functors $\mathcal{C} \to \mathrm{Sp}$ in a systematic way.

Let $\mathsf{Fun}_*(\mathcal{C}, \mathrm{Sp})$ be the ∞-category of functors $F : \mathcal{C} \to \mathrm{Sp}$ that are

- reduced: $F(*) \simeq *$
- finitary: F preserves filtered colimits

Day convolution

There are symmetric monoidal ∞-categories:

- Sp: with the smash product $\wedge$
- $\mathsf{Fun}_*(\mathcal{C}, \mathsf{Sp})$: with the objectwise smash product $\bar{\wedge}$ of functors:

$$(F \bar{\wedge} G)(X) = F(X) \wedge G(X).$$

There is a symmetric monoidal structure $\otimes$ on the ∞-category of functors

$$\mathsf{Fun}_*(\mathcal{C}, \mathsf{Sp}) \to \mathsf{Sp}$$

that admits natural equivalences (of spectra)

$$\mathsf{Nat}(\mathsf{A} \otimes \mathsf{B}, \mathsf{C}) \simeq \mathsf{Nat}_{F,G}(\mathsf{A}(F) \wedge \mathsf{B}(G), \mathsf{C}(F \bar{\wedge} G)).$$

This is the Day convolution of A and B. (This construction is due to Glasman in the ∞-categorical context.)

An Example of Day convolution

Example

For object $X \in \mathcal{C}$ we have the *evaluation functor*

$$\mathrm{ev}_X : \mathrm{Fun}_*(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}; \quad \mathrm{ev}_X(F) = F(X).$$

What is $\mathrm{ev}_X \otimes \mathrm{ev}_Y$?

By Yoneda (for Sp-valued functors), ev_X is representable:

$$\mathrm{ev}_X(F) \simeq \mathrm{Nat}(R_X, F); \quad R_X(-) := \Sigma^\infty \mathrm{Hom}_{\mathcal{C}}(X, -)$$

The Day convolution of representable functors is representable, so:

$$\begin{aligned}(\mathrm{ev}_X \otimes \mathrm{ev}_Y)(F) &\simeq \mathrm{Nat}(R_X \barwedge R_Y, F) \\ &\simeq \dots \\ &\simeq \mathrm{cr}_2 F(X, Y).\end{aligned}$$

Day convolution and cross-effects

When F is reduced

$$\mathrm{ev}_X(F) = F(X) \simeq \mathrm{cr}_1(F)(X)$$

so the previous example can be wrtten

$$\mathrm{cr}_1(-)(X) \otimes \mathrm{cr}_1(-)(Y) \simeq \mathrm{cr}_2(-)(X, Y).$$

More generally:

Lemma

For $X_1, \dots, X_n \in \mathcal{C}$:

$$\begin{aligned} \mathrm{cr}_n(-)(X_1, \dots, X_n) &\simeq \mathrm{cr}_1(-)(X_1) \otimes \cdots \otimes \mathrm{cr}_1(-)(X_n) \\ &\simeq \mathrm{ev}_{X_1} \otimes \cdots \otimes \mathrm{ev}_{X_n}. \end{aligned}$$

Polynomial Functors

Definition

A functor $F : \mathcal{C} \to \mathrm{Sp}$ is *n-excisive* if it takes strongly cocartesian $(n+1)$-cubes to pullback cubes.

Construction (Goodwillie)

Any $F : \mathcal{C} \to \mathrm{Sp}$ has a Taylor tower:

$$F \to \cdots \to P_n F \to P_{n-1} F \to \dots P_1 F \to *$$

where $F \to P_n F$ is the universal approximation to F by an n-excisive functor.

Layers and Derivatives

Theorem (Goodwillie)

The layer $D_nF := \mathrm{fib}(P_nF \to P_{n-1}F)$ is n-homogeneous and is given in general by

$$D_nF(X) \simeq \partial_nF(X, \ldots, X)_{\Sigma_n}$$

where $\partial_nF : \mathcal{C}^n \to \mathrm{Sp}$ is symmetric multilinear.

Example

When $\mathcal{C} = \mathrm{Top}_*$, the functor ∂_nF is determined by a single spectrum E_n (the n^{th} derivative of F):

$$\partial_nF(X_1, \ldots, X_n) \simeq E_n \wedge X_1 \wedge \ldots \wedge X_n.$$

Main Question

How can we reconstruct F (or at least its Taylor tower) from its layers, i.e. from the collection $\partial_*F = (\partial_nF)_{n\geq 1}$?

Day convolution and derivatives

For $X_1, \ldots, X_n \in \mathcal{C}$ there is a functor

$$\partial_n(-)(X_1, \ldots, X_n) : \mathrm{Fun}_*(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}.$$

Theorem (C.)

For a pointed compactly-generated ∞-category $\mathcal{C}$, there is an equivalence (of functors $\mathrm{Fun}_*(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Sp}$):

$$\partial_n(-)(X_1, \ldots, X_n) \simeq \partial_1(-)(X_1) \otimes \cdots \otimes \partial_1(-)(X_n)$$

natural in $X_1, \ldots, X_n \in \mathcal{C}$.

Proof.

The n^{th} derivative can be written as a colimit of (desuspended) cross-effects. Then apply the previous calculation. □

Operads

Let $\mathcal{C}$ be a pointed compactly-generated ∞-category.

Construction

There is a stable ∞-operad $\mathbb{I}_{\mathcal{C}}$ (think: coloured operad, enriched in Sp*) with*

- *colours = objects of $\mathcal{C}$;*
- *multi-morphism spectra $\mathbb{I}_{\mathcal{C}}(X_1, \dots, X_n; Y)$ given by:*

$$\mathrm{Nat}(\partial_1(-)(Y)\ ,\ \partial_1(-)(X_1) \otimes \cdots \otimes \partial_1(X_n))$$

This is a coendomorphism operad for the objects $\partial_1(-)(X)$ (for $X \in \mathcal{C}$) under the symmetric monoidal structure of Day convolution.

This construction uses work of Barwick-Glasman-Nardin on the opposite of a symmetric monoidal ∞-category.

Module structure on derivatives

For $F : \mathcal{C} \to \mathrm{Sp}$, we have maps

$$\begin{array}{c}
\partial_k F(Y_1, \ldots, Y_k) \wedge \mathbb{I}_{\mathcal{C}}(\underline{X}_1; Y_1) \wedge \ldots \wedge \mathbb{I}_{\mathcal{C}}(\underline{X}_k; Y_k) \\
\downarrow \simeq \\
\partial_1^{\otimes k}(F) \wedge \mathrm{Nat}(\partial_1, \partial_1^{\otimes n_1}) \wedge \ldots \wedge \mathrm{Nat}(\partial_1, \partial_1^{\otimes n_k}) \\
\downarrow \\
\partial_1^{\otimes n_1 + \cdots + n_k}(F) \\
\downarrow \simeq \\
\partial_n F(\underline{X}_1, \ldots, \underline{X}_k)
\end{array}$$

that make the symmetric sequence $\partial_* F = (\partial_n F)_{n \geq 1}$ into a right module over the ∞-operad $\mathbb{I}_{\mathcal{C}}$.

Derivatives of the identity functor

Any functor $G : \mathcal{C} \to \mathcal{D}$ between pointed compactly-generated ∞-categories has derivatives

$$\partial_n G(X_1, \dots, X_n; Y)$$

for $X_1, \dots, X_n \in \mathcal{C}$ and $Y \in \mathcal{D}^{op}$.

Theorem (C.)

The terms in the operad $\mathbb{I}_{\mathcal{C}}$ can be identified with the derivatives of the identity functor $I_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$:

$$\begin{aligned}\mathbb{I}_{\mathcal{C}}(X_1, \dots, X_n; Y) &\simeq \mathrm{Nat}(\partial_1(-)(Y), \partial_n(-)(X_1, \dots, X_n)) \\ &\simeq \partial_n I_{\mathcal{C}}(X_1, \dots, X_n; Y).\end{aligned}$$

So, for $F : \mathcal{C} \to \mathrm{Sp}$, $\partial_* F$ is a right module over $\partial_* I_{\mathcal{C}}$. But this is not enough structure to be able to reconstruct the Taylor tower of F.

Pro-operads

The first derivative is a colimit:

$$\partial_1 F \simeq P_1 F \simeq \mathrm{colim}(F \to \Omega F \Sigma \to \Omega^2 F \Sigma^2 \to \dots).$$

Hence

$$\mathbb{I}_{\mathcal{C}}(n) \simeq \mathrm{Nat}(\partial_1, \partial_1^{\otimes n}) \simeq \lim_L \mathrm{Nat}(\Omega^L(-)\Sigma^L, \partial_1^{\otimes n}).$$

So write

$$\mathbb{I}_{\mathcal{C}}^L(n) = \mathrm{Nat}(\Omega^L(-)\Sigma^L, \partial_1^{\otimes n}).$$

The inverse system

$$\mathbb{I}_{\mathcal{C}}^\bullet := (\mathbb{I}_{\mathcal{C}}^L)_{L \geq 0}$$

is a pro-operad (a monoid in the category of pro-symmetric sequences).

Modules over a pro-operad

A right module over the operad $\mathbb{I}_{\mathcal{C}}$ consists of a symmetric sequence A and suitable maps

$$A(k) \to \mathrm{Map}(\mathbb{I}_{\mathcal{C}}(n_1) \wedge \ldots \wedge \mathbb{I}_{\mathcal{C}}(n_k), A(n))^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}.$$

A right module over the pro-operad $\mathbb{I}^{\bullet}_{\mathcal{C}}$ consists of maps

$$A(k) \to \operatorname*{colim}_{L} \mathrm{Map}(\mathbb{I}^{L}_{\mathcal{C}}(n_1) \wedge \ldots \wedge \mathbb{I}^{L}_{\mathcal{C}}(n_k), A(n))^{\Sigma_{n_1} \times \cdots \times \Sigma_{n_k}}.$$

For $F : \mathcal{C} \to \mathrm{Sp}$, the right $\mathbb{I}_{\mathcal{C}}$-module structure on $\partial_* F$ can be refined to a right $\mathbb{I}^{\bullet}_{\mathcal{C}}$-module structure.

Theorem (C.)

Let $\mathcal{C}$ be a pointed compactly-generated ∞-category. Then there is an equivalence of ∞-categories

$$\partial_* : (n\text{-excisive functors } \mathcal{C} \to \mathrm{Sp}) \xrightarrow{\sim} (n\text{-truncated right } \mathbb{I}^{\bullet}_{\mathcal{C}}\text{-modules}).$$